

A POLISH GROUP CONTAINING A HAAR NULL F_σ -SUBGROUP THAT CANNOT BE ENLARGED TO A HAAR NULL G_δ -SET

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ABSTRACT. Answering a question of Elekes and Vidnyánszky, we construct a Polish meta-abelian group H and a subgroup $F \subset H$, which is a Haar null F_σ -set in H that cannot be enlarged to a Haar null G_δ -set.

A Borel subset B of a Polish group H is called *Haar null* if there exists a σ -additive Borel probability measure μ on H such that $\mu(xBy) = 0$. It is well-known [1] that a Borel subset of a Polish locally compact group H is Haar null if and only if B has Haar measure zero if and only if B is contained in a Haar null G_δ -subset of H . As was shown by Elekes and Vidnyánszky [2], for non-locally compact Polish groups the latter result is not true: each non-locally compact Polish abelian group H contains a Borel Haar null subset $B \subset H$ which is not contained in a Haar null G_δ -set in H . However, the construction of the Borel set B exploited in [2] does not allow to evaluate the Borel complexity of B . Because of that, Elekes and Vidnyánszky asked in [2] whether each Haar null F_σ -subset B of a Polish (Abelian) group H can be enlarged to a Haar null G_δ -set. In this section we partly answer this problem presenting an example of a Polish meta-Abelian group H and a Haar null F_σ -subgroup $F \subset H$ that cannot be enlarged to a Haar null G_δ -set.

A topological group H is *meta-abelian* if H contains a closed normal abelian subgroup $A \subset H$ such that the quotient group H/A is Abelian.

We define a subset B of topological group H to be *thick* if for every compact subset $K \subset H$ there are points $x, y \in H$ such that $xKy \subset B$. It is easy to see that a thick Borel subset of Polish groups cannot be Haar null.

Theorem 1. *There exists a Polish meta-abelian group H containing a subgroup $F \subset H$ such that F is a Haar null F_σ -set in H but every G_δ -set $G \subset H$ containing B is thick and hence is not Haar null in H .*

Proof. Observe that the countable power $R = \mathbb{R}^\omega$ of the real lines has the structure of a unital topological ring and the dense G_δ -set $(\mathbb{R} \setminus \{0\})^\omega$ coincides with the set R^* of invertible elements of the ring R . On the product $H = R \times R^*$ consider the binary operation $\star : H \times H \rightarrow H$ defined by the formula

$$(x, a) \star (y, b) = (x + ay, ab) \quad \text{for } (x, a), (y, b) \in H = R \times R^*.$$

The Polish space $H = R \times R^*$ endowed with this binary operation is a Polish group called the *semidirect product* of R and R^* .

In the topological ring $R = \mathbb{R}^\omega$ consider the F_σ -subset

$$R_0 = \{(x_n)_{n \in \omega} \in \mathbb{R}^\omega : \exists n \in \omega \ \forall m \geq n \ x_m = 0\}$$

consisting of all eventually zero sequences. It follows that R_0 is a subring of R and $F = R_0 \times R^* \subset H$ is a subgroup of the Polish group H . We claim that the subgroup F has the desired properties.

Lemma 1. *The subset F is Haar null in H .*

Proof. Taking into account that R_0 is a Borel non-open subgroup of the Polish Abelian group R , we can apply a classical result of Christensen [1] and conclude that R_0 is Haar null in R . This allows us to find a probability measure μ on R such that $\mu(x + R_0) = 0$ for all $x \in R$. Identifying R with the normal subgroup $R \times \{1\}$ of H , we can consider the measure μ as a measure on H . Then for every elements $(x, a), (y, b) \in H$ we get

$$((x, a) \star F \star (y, b)) \cap (R \times \{1\}) = (x, a) \star (R_0 \times \{a^{-1}b^{-1}\}) \star (y, b) = (x + a \cdot R_0 + a^{-1}b^{-1}y, 1)$$

and hence

$$\mu((x, a) \star F \star (y, b)) = \mu(x + a \cdot R_0 + a^{-1}b^{-1}y) = \mu(x + a^{-1}b^{-1}y + R_0) = 0$$

by the choice of μ . So, the measure μ witnesses that the set $F = R_0 \times R^*$ is Haar null in H . □

Lemma 2. *Every G_δ -set $G \subset H$ containing F is thick.*

Proof. Given a G_δ -set $G \subset H$ containing F , consider its complement $H \setminus G$ and write it as a countable union $H \setminus G = \bigcup_{k \in \omega} E_k$ of an increasing sequence $(E_k)_{k \in \omega}$ of closed sets in H . To prove that G is thick in H , it suffices for every compact subset $K \subset H$ to find an element $h \in H$ such that $hKh^{-1} \subset G$. Given a compact set $K \subset H$ choose compact sets $C \subset R$ and $C^* \subset R^*$ such that $K \subset C \times C^*$. Observe that for every element $a \in R^*$ and the element $h = (0, a) \in H$ we get $h^{-1} = (0, a^{-1})$ and hence

$$hKh^{-1} \subset (0, a) \star (C \times C^*) \star (0, a^{-1}) \subset aC \times C^*.$$

Now it suffices to find an element $a \in R^*$ such that $(aC \times C^*) \cap E_k = \emptyset$ for all $k \in \omega$. The compactness of the set C^* guarantees that the projection $\text{pr} : R \times C^* \rightarrow R$, $\text{pr} : (x, y) \mapsto x$, is closed and for every $k \in \omega$ the set $P_k = \text{pr}(E_k \cap (R \times C^*))$ is closed in R and does not intersect the F_σ -subgroup R_0 . Observe that the set $R_k^* = \{a \in R^* : aC \cap P_k = \emptyset\}$ is contained in $\{a \in R^* : (aC \times C^*) \cap E_k \neq \emptyset\}$. The compactness of the set C implies that the set R_n^* is open in R^* .

Claim 1. *For every $k \in \omega$ the set R_k^* is dense in R^* .*

Proof. Given any element $a = (a_n)_{n \in \omega} \in R = \mathbb{R}^\omega$ and any neighborhood $O_a \subset R$, find $m \in \omega$ such that the closed subspace $\{(b_n)_{n \in \omega} \in \mathbb{R}^\omega : \forall n < m \ b_n = a_n\}$ is contained in O_a . Find a sequence $(C_n)_{n \in \omega}$ of compact subsets of the real line such that $C \subset \prod_{n \in \omega} C_n$. Consider the compact space $\Pi = \prod_{n < k} a_n C_n$ and observe that the projection $\pi : \Pi \times \mathbb{R}^{\omega \setminus k} \rightarrow \mathbb{R}^{\omega \setminus k}$, $\pi : (x, y) \mapsto y$, is a closed map. This implies that the set $\tilde{P}_k = \pi_k((\Pi \times \mathbb{R}^{\omega \setminus k}) \cap P_k)$ is closed in $\mathbb{R}^{\omega \setminus k}$ and does not contain the constant zero function $z \in \mathbb{R}^{\omega \setminus k}$. By the compactness of the product $C_{\geq k} = \prod_{n \geq k} C_n \subset \mathbb{R}^{\omega \setminus k}$ and the continuity of multiplication in the topological ring $\mathbb{R}^{\omega \setminus k}$, there exists an element $\varepsilon \in (\mathbb{R} \setminus \{0\})^{\omega \setminus k}$ so close to zero that $(\varepsilon \cdot C_{\geq k}) \cap \tilde{P}_k = \emptyset$. Then for the element $b = (b_n)_{n \in \omega} \in O_a$ defined by $b_n = a_n$ for $n < k$ and $b_n = \varepsilon_n$ for $n \geq k$ we get

$$(b \cdot C) \cap P_n \subset \left(\prod_{n \in \omega} b_n C_n \right) \cap P_n \subset \left(\prod_{n < k} a_n C_n \right) \times \left(\left(\prod_{n \geq k} \varepsilon_n C_n \right) \cap \tilde{P}_k \right) = \emptyset$$

and hence $b \in O_a \cap R_k^*$. □

Claim 1 combined with the Baire Theorem guarantees that $\bigcap_{k \in \omega} R_k^*$ is a dense G_δ -set in the Polish space R^* . Then we can take any point $a \in \bigcap_{k \in \omega} R_k^*$ and conclude that $(aC \times C^*) \cap \bigcup_{k \in \omega} E_k = \emptyset$ and hence for the element $h = (0, a)$ we get $hKh^{-1} \subset G$. □

□

However, Theorem 1 gives no answer to the following two problems posed by Elekes and Vidnyánszky in [2].

Problem 1. *Is each Haar null F_σ -subset of an uncountable Polish Abelian group G contained in a Haar null G_δ -subset of G ?*

Problem 2. *Is each countable subset of an uncountable Polish group G contained in a Haar null G_δ -subset of G ?*

By Remark 5.3 [2] the answer to Problem 2 is affirmative for Polish abelian groups.

REFERENCES

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